Answer Key for Ch14 Exercise1: Derive $\hat{\beta}$

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1. Apply the logic developed in this chapter to the model $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$. Derive the OLS estimate for $\hat{\beta}_0$ and $\hat{\beta}_1$.

   (a) Write out the sum of squared residuals for the model

$$\sum \hat{\epsilon}_i^2 \;=\; \sum (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2$$

   (b) We minimize the sum of squared residuals by taking the derivatives with respect to $\hat{\beta}_0$ (one equation) and $\hat{\beta}_1$ (a second equation) and setting them to zero.[1] The estimates are the values which produce a derivative of zero.

$$\begin{aligned}
\frac{\partial \sum \hat{\epsilon}_i^2}{\partial \hat{\beta}_0} &= (-2)\sum (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) = 0 \\
\frac{\partial \sum \hat{\epsilon}_i^2}{\partial \hat{\beta}_1} &= (-2)\sum (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)(X_i) = 0
\end{aligned}$$

   (c) Solving for $\hat{\beta}_0$ is relatively straightforward. Divide both sides by (-2) and do the following steps:

$$\begin{aligned}
\sum Y_i - \sum \hat{\beta}_0 - \sum \hat{\beta}_1 X_i &= 0 \text{ Separate the sum into components} \\
\sum Y_i - \sum \hat{\beta}_1 X_i &= \sum \hat{\beta}_0 \text{ Move } \hat{\beta}_0 \text{ to RHS} \\
\sum Y_i - \hat{\beta}_1 \sum X_i &= N\hat{\beta}_0 \text{ Pull constants out of summations} \\
\frac{\sum Y_i}{N} - \hat{\beta}_1 \frac{\sum X_i}{N} &= \hat{\beta}_0 \text{ Divide by } N \\
\overline{Y} - \hat{\beta}_1 \overline{X} &= \hat{\beta}_0 \text{ Use definition of mean}
\end{aligned}$$

   (d) To solve for $\hat{\beta}_1$ divide both sides of the derivative with respect to $\hat{\beta}_1$ by (-2)

[1] These are often referred to as "normal equations" (as if we haven't used the word *normal* enough in statistics).

and do the following steps:

$$\begin{aligned}
\sum Y_iX_i - \sum \hat{\beta}_0 X_i - \sum \hat{\beta}_1 X_i^2 &= 0 \text{ Separate the sum into components} \\
\sum Y_iX_i - \sum (\overline{Y} - \hat{\beta}_1\overline{X})X_i - \sum \hat{\beta}_1 X_i^2 &= 0 \text{ Substitute for } \hat{\beta}_0 \\
\sum Y_iX_i - \sum \overline{Y}X_i &= \hat{\beta}_1 (\sum X_i^2 - \overline{X}\sum X_i) \text{ Simplify} \\
\hat{\beta}_1 &= \frac{\sum Y_iX_i - \sum \overline{Y}X_i}{\sum X_i^2 - \overline{X}\sum X_i} \text{ Solve for } \hat{\beta}_1
\end{aligned}$$

(e) We prefer the equation for $\hat{\beta}_1$ to be in the more intuitive mean-deviated form.

i. First substitute for $\sum X_i = N\overline{X}$. Be careful to note that $\sum Y_iX_i$ cannot be simplified; we can only use this for terms where the only thing in the summation is X or X times a constant.

$$\hat{\beta}_1 = \frac{\sum Y_iX_i - N\overline{Y}\overline{X}}{\sum X_i^2 - N\overline{X}^2}$$

ii. Next use facts that $\sum X_i = N\overline{X}$ and $\sum Y_i = N\overline{Y}$ to rewrite $(Y_i - \overline{Y})(X_i - \overline{X})$, which is the form we are seeking for the numerator of the $\hat{\beta}_1$ equation.

$$\begin{aligned}
\sum (Y_i - \overline{Y})(X_i - \overline{X}) &= \sum Y_iX_i - \overline{X}\sum Y_i - \overline{Y}\sum X_i + N\overline{Y}\overline{X} \\
&= \sum Y_iX_i - N\overline{X}\overline{Y} - N\overline{Y}\overline{X} + N\overline{Y}\overline{X} \\
&= \sum Y_iX_i - N\overline{X}\overline{Y}
\end{aligned}$$

which is the value we had in the numerator of $\hat{\beta}_1$ in part (d) above, meaning we can use $\sum (Y_i - \overline{Y})(X_i - \overline{X})$ in the numerator.

iii. Do similar steps for the denominator of the $\hat{\beta}_1$ equation.

$$\begin{aligned}
\sum (X_i - \overline{X})(X_i - \overline{X}) &= \sum X_i^2 - 2\overline{X}\sum X_i + N\overline{X}^2 \\
&= \sum X_i^2 - 2N\overline{X}^2 + N\overline{X}^2 \\
&= \sum X_i^2 - N\overline{X}^2
\end{aligned}$$

which is the value we had in the denominator of $\hat{\beta}_1$ meaning we can use $\sum (X_i - \overline{X})^2$ in the denominator, meaning that

$$\hat{\beta}_1 = \frac{\sum (Y_i - \overline{Y})(X_i - \overline{X})}{\sum (X_i - \overline{X})^2}$$