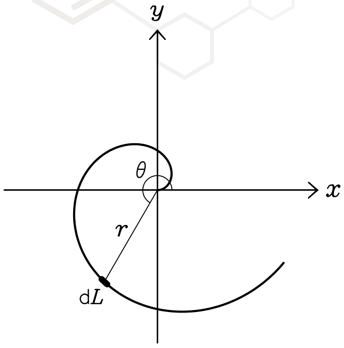
11 Line integrals



> **(1)** Calculate the length of the spiral specified in polar coordinates by $r = \theta$, $0 \leqslant \theta \leqslant \theta_{\text{max}}$ radians.

The Cartesian coordinates (x, y) of the point specified by the polar parameters r and θ are given by

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

Infinitesimally small changes in the polar parameters, $\mathrm{d}r$ and $\mathrm{d}\theta$, lead to corresponding tiny changes in the Cartesian coordinates

$$\mathrm{d}x = \left(\frac{\partial x}{\partial r}\right)_\theta \mathrm{d}r + \left(\frac{\partial x}{\partial \theta}\right)_r \mathrm{d}\theta = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta\,, \quad \text{and}$$

$$\mathrm{d}y = \left(\frac{\partial y}{\partial r}\right)_\theta \mathrm{d}r + \left(\frac{\partial y}{\partial \theta}\right)_r \mathrm{d}\theta = \sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta$$

By Pythagoras theorem, therefore, the tiny arc length, $\mathrm{d}L$, between two neighbouring points along the path $r = r(\theta)$ is

$\sin^2\theta + \cos^2\theta = 1$

$$\mathrm{d}L^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + (r\,\mathrm{d}\theta)^2$$

Hence, for $r = \theta$, the length of the spiral is given by

$\cosh^2 u = 1 + \sinh^2 u$

$$L = \int_{\theta=0}^{\theta=\theta_{\text{max}}} \mathrm{d}L = \int_0^{\theta_{\text{max}}} \sqrt{1+\theta^2}\;\mathrm{d}\theta$$

$\mathrm{d}\theta = \cosh u\;\mathrm{d}u$

which can be solved by substituting $\theta = \sinh u$:

$\cosh 2u = 2\cosh^2 u - 1$

$$L = \int_0^{u_{\text{max}}} \cosh^2 u\;\mathrm{d}u = \frac{1}{2}\int_0^{u_{\text{max}}} \big(1 + \cosh 2u\big)\,\mathrm{d}u = \frac{1}{2}\Big[u + \frac{1}{2}\sinh 2u\Big]_0^{u_{\text{max}}}$$

where $u_{\text{max}} = \sinh^{-1}\theta_{\text{max}}$. Finally, with the aid of hyperbolic function identities, the required length reduces to

$\sinh 2u = 2\sinh u\cosh u$

$$L = \frac{1}{2}u_{\text{max}} + \frac{1}{4}\sinh(2u_{\text{max}}) = \underline{\frac{1}{2}\Big(\sinh^{-1}\theta_{\text{max}} + \theta_{\text{max}}\sqrt{1+\theta_{\text{max}}^2}\Big)}$$