6 Taylor series

(1) Derive the Taylor series for $\sin(x+\pi/6)$ for small x.

$$
\begin{aligned}
\text{Let}\quad f(x) &= \sin x & \therefore\quad f(\pi/6) &= 1/2\\
\therefore\quad f'(x) &= \cos x & f'(\pi/6) &= \sqrt{3}/2\\
f''(x) &= -\sin x & f''(\pi/6) &= -1/2\\
f'''(x) &= -\cos x & f'''(\pi/6) &= -\sqrt{3}/2\\
f''''(x) &= \sin x & f''''(\pi/6) &= 1/2
\end{aligned}
$$

$$\text{But}\quad f(x+a) = f(a) + x\,f'(a) + \frac{x^2}{2!}\,f''(a) + \frac{x^3}{3!}\,f'''(a) + \cdots$$

$$\therefore\quad \underline{\sin\left(x+\tfrac{\pi}{6}\right) = \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\,x - \tfrac{1}{4}\,x^2 - \tfrac{\sqrt{3}}{12}\,x^3 + \tfrac{1}{48}\,x^4 + \cdots}$$

This Taylor series could also be ascertained by expanding $\sin(x+\pi/6)$ with the compound-angle formula for sines

$$\sin\left(x+\tfrac{\pi}{6}\right) = \sin x\,\cos\left(\tfrac{\pi}{6}\right) + \cos x\,\sin\left(\tfrac{\pi}{6}\right) = \tfrac{\sqrt{3}}{2}\sin x + \tfrac{1}{2}\cos x$$

and then using the well-known series for $\sin x$ and $\cos x$:

$$
\begin{aligned}
\therefore\quad \sin\left(x+\tfrac{\pi}{6}\right) &= \tfrac{\sqrt{3}}{2}\left(x - \tfrac{1}{6}\,x^3 + \cdots\right) + \tfrac{1}{2}\left(1 - \tfrac{1}{2}\,x^2 + \tfrac{1}{24}\,x^4 - \cdots\right)\\
&= \underline{\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\,x - \tfrac{1}{4}\,x^2 - \tfrac{\sqrt{3}}{12}\,x^3 + \tfrac{1}{48}\,x^4 + \cdots}
\end{aligned}
$$