Multiple Random Variables 4

4.B Appendix: The Capital Asset Pricing Model

Calculation workbooks: `three_assets.xlsx`
`three_plus_one.xlsx`
`Sharpe_ratio.xlsx`
`security_market_line.xlsx`

U.S. stock and bond returns, 1926–2002.

We begin our study of the relationship between risk and return by examining data on the annual returns of various investment categories from 1926 to 2002.[1]

investment	average	standard deviation
small-company stocks	16.9%	33.2%
large-company stocks	12.2%	20.5%
long-term corporate bonds	6.2%	8.7%
long-term government Bonds	5.8%	9.4%
U.S. Treasury bills	3.8%	3.2%

There's a striking relationship in this data: investment classes with the highest returns are also those whose returns fluctuate most.

We commonly use the word **risk** to refer to various aspects of return variability. The data in the table of annual investment returns suggests a connection between an asset's expected return and the variability of its return. In this appendix, we investigate the exact nature of this relationship. In brief, we argue that investors should care about two distinct types of risk—*absolute risk* and *marginal risk*—and that only the holding of marginal risk is associated with high expected returns.

[1]Roger Ibbotson and Rex Sinquefield, *Stocks, Bonds, Bills, and Inflation: 2003 Yearbook*, Chicago: Ibbotson Associates.

The theory we describe here is known as the *capital asset pricing model*, or *CAPM* for short. The CAPM is enormously influential among financial professionals, not only for portfolio evaluation, but also for capital budgeting (i.e., the management of assets within an individual firm) and other valuation tasks. The final product of the CAPM is a formula that relates risk and expected returns, where levels of risks are quantified by measurements called *betas*. Betas are now part of the basic lexicon of portfolio choice, and the betas of a vast array of stocks and mutual funds can be looked up on any comprehensive financial services website. The vast majority of firms utilize the CAPM when making capital budgeting decisions: 74% of firms always or almost always use the CAPM during cash flow analyses of new projects.[2]

The ideas we present in this chapter were first developed in the 1950s and 1960s, with seminal contributions by Harry Markowitz, William F. Sharpe, and John Lintner.[3] In 1990, Markowitz and Sharpe (along with Merton Miller) were awarded the Nobel Prize for their pioneering work in financial economics.

4.B.1 Portfolio selection with many risky assets

In the financial market model from Section 4.4, investors allocated their capital between two risky assets. In this section, we extend this model to allow for investment in many risky assets; in the next, we introduce the possibility of investment in a riskless asset.

Portfolios with many risky assets.

In our new model, investors allocate their capital among n assets with uncertain returns. The (per-dollar) **returns** on these assets are described by the random variables $R_1, R_2, \ldots, R_n$. The means, variances, and covariances of these returns are denoted $\mu_i = \mathrm{E}(R_i)$, $\sigma_i^2 = \mathrm{Var}(R_i)$, and $\sigma_{i,j} = \mathrm{Cov}(R_i, R_j)$, respectively.

An investor's allocation is described by a **portfolio**, which is a list of numbers $p = (p_1, p_2, \ldots, p_n)$ describing the percentage of his investment capital placed in each of the individual assets. By definition, the portfolio weights p_i must add up to 100%: $\sum_{i=1}^{n} p_i = 1$.

The percentage return on the portfolio p, which we denote by R_p, is the weighted average of the asset returns R_i, with weights given by the portfolio weights p_i:

$$R_p = \sum_{i=1}^{n} p_i R_i.$$

[2] John R. Graham and Campbell R. Harvey, "The Theory and Practice of Corporate Finance: Evidence from the Field," *Journal of Financial Economics* 60 (2001), 187–243.

[3] Their key papers include: Harry Markowitz, "Portfolio Selection," *Journal of Finance* 7 (1952), 77–91; William F. Sharpe, "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk," *Journal of Finance* 19 (1964), 425–442; and John Lintner, "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," *Review of Economics and Statistics* 47 (1965), 13–37.

The portfolio returns R_p are a function of the random variables $R_1, \ldots, R_n$, and R_p is thus a random variable itself.

To determine the mean return and the variance of returns for portfolio p, we use formulas for the traits of the sum of n random variables. The first formula,

$$\mathrm{E}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{E}(X_i), \tag{4.13}$$

says that the mean of a sum equals the sum of the means, just as in the two asset case. The second formula, for the variance of a sum, is more complicated. We state it here in three equivalent forms.

$$\begin{aligned} \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{Cov}(X_i, X_j) \\ &= \sum_{i=1}^{n} \mathrm{Var}(X_i) + \sum_{i=1}^{n}\sum_{j\neq i} \mathrm{Cov}(X_i, X_j) \\ &= \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2\sum_{i=1}^{n-1}\sum_{j>i} \mathrm{Cov}(X_i, X_j). \end{aligned} \tag{4.14}$$

To understand the first version of (4.14), draw a table with n rows and columns. Fill in the table with entries of the form $\mathrm{Cov}(X_i, X_j)$, where i is the row number and j the column number (Figure 4.B.1(i)). The first expression for the variance of $X_1 + \cdots + X_n$ is the sum of the n^2 entries in this table. Using formula (4.7) to replace each diagonal term $\mathrm{Cov}(X_i, X_i)$ with $\mathrm{Var}(X_i)$, we obtain the second version of (4.14). Then, using formula (4.6) to replace each expression $\mathrm{Cov}(X_i, X_j)$ with $i > j$ (i.e., each expression southwest of the diagonal) with $\mathrm{Cov}(X_j, X_i)$, we find that each of these terms has a twin to the northeast of the diagonal (Figure 4.B.1(ii)). The third version of (4.14) describes the sum of the table entries in this new table: namely, the sum of the variance terms on the diagonal, plus twice the sum of the covariance terms northeast of the diagonal. Since this final version of the formula has the fewest terms, it is the fastest one for calculations.

When $n = 2$, our table has four entries, two variances and two covariances (Figure 4.B.1(iii)), and we obtain our old formula for the variance of the sum of two random variables:

$$\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 2\,\mathrm{Cov}(X_1, X_2).$$

The more random variables we sum, the higher the proportion of entries that are covariances: of the n^2 terms, n are variances, so the remaining $n^2 - n$ are covariances. For example, if we sum 100 random variables, the variance of the sum has 10,000 terms; 100 are variances, and 9900 covariances. As we shall see, the fact that the formula for the variance of a sum of many random variables is dominated by covariances has deep implications for the pricing of financial assets.

	1	2	3	$\cdots$	n
1	$\text{Cov}(X_1, X_1)$	$\text{Cov}(X_1, X_2)$	$\text{Cov}(X_1, X_3)$	$\cdots$	$\text{Cov}(X_1, X_n)$
2	$\text{Cov}(X_2, X_1)$	$\text{Cov}(X_2, X_2)$	$\text{Cov}(X_2, X_3)$		
3	$\text{Cov}(X_3, X_1)$	$\text{Cov}(X_3, X_2)$	$\text{Cov}(X_3, X_3)$		
$\vdots$	$\vdots$			$\ddots$	
n	$\text{Cov}(X_n, X_1)$	$\cdots$	$\cdots$	$\cdots$	$\text{Cov}(X_n, X_n)$

(i)

	1	2	3	$\cdots$	n
1	$\text{Var}(X_1)$	$\text{Cov}(X_1, X_2)$	$\text{Cov}(X_1, X_3)$	$\cdots$	$\text{Cov}(X_1, X_n)$
2	$\text{Cov}(X_1, X_2)$	$\text{Var}(X_2)$	$\text{Cov}(X_2, X_3)$		
3	$\text{Cov}(X_1, X_3)$	$\text{Cov}(X_2, X_3)$	$\text{Var}(X_3)$		
$\vdots$	$\vdots$			$\ddots$	
n	$\text{Cov}(X_1, X_n)$	$\cdots$	$\cdots$	$\cdots$	$\text{Var}(X_n)$

(ii)

	1	2
1	$\text{Cov}(X_1, X_1)$ $= \text{Var}(X_1)$	$\text{Cov}(X_1, X_2)$
2	$\text{Cov}(X_2, X_1)$ $= \text{Cov}(X_1, X_2)$	$\text{Cov}(X_2, X_2)$ $= \text{Var}(X_2)$

(iii)

Figure 4.B.1: Calculating the variance of a sum.

Combining formulas (4.13) and (4.14) with formulas (4.8)–(4.10) for the traits of multiples of random variables, we compute the mean returns and variance of returns on portfolio p:

$$\text{E}(R_p) = \text{E}\left(\sum_{i=1}^{n} p_i R_i\right) = \sum_{i=1}^{n} \text{E}(p_i R_i) = \sum_{i=1}^{n} p_i \text{E}(R_i) = \sum_{i=1}^{n} p_i \mu_i,$$

$$\begin{aligned}
\text{Var}(R_p) &= \text{Var}\left(\sum_{i=1}^{n} p_i R_i\right) \\
&= \sum_{i=1}^{n} \text{Var}(p_i R_i) + 2 \sum_{i=1}^{n-1} \sum_{j>i} \text{Cov}(p_i R_i, p_j R_j) \\
&= \sum_{i=1}^{n} p_i^2 \, \text{Var}(R_i) + 2 \sum_{i=1}^{n-1} \sum_{j>i} p_i p_j \, \text{Cov}(R_i, R_j) \\
&= \sum_{i=1}^{n} p_i^2 \sigma_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j>i} p_i p_j \sigma_{i,j}.
\end{aligned}$$

■ Example An investor is choosing a portfolio consisting of stock in three corporations, Acme, Bake, and Colic. The means, variances, and covariances of the returns on these stocks are described by the **mean vector** and **covariance matrix** below:

$$\begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix} = \begin{pmatrix} .10 \\ .20 \\ .30 \end{pmatrix}, \quad \begin{pmatrix} \sigma_A^2 & \sigma_{A,B} & \sigma_{A,C} \\ \sigma_{B,A} & \sigma_B^2 & \sigma_{B,C} \\ \sigma_{C,A} & \sigma_{C,B} & \sigma_C^2 \end{pmatrix} = \begin{pmatrix} .0064 & .0036 & .0012 \\ .0036 & .0144 & .0036 \\ .0012 & .0036 & .0225 \end{pmatrix}.$$

The standard deviations and correlations of the assets' returns are therefore

$$\sigma_A = \sqrt{\sigma_A^2} = .08 \qquad \sigma_B = \sqrt{\sigma_B^2} = .12 \qquad \sigma_C = \sqrt{\sigma_C^2} = .15$$

$$\rho_{A,B} = \frac{\sigma_{A,B}}{\sigma_A \sigma_B} = .375 \qquad \rho_{A,C} = \frac{\sigma_{A,C}}{\sigma_A \sigma_C} = .1 \qquad \rho_{B,C} = \frac{\sigma_{B,C}}{\sigma_B \sigma_C} = .2.$$

Suppose the investor holds the portfolio $p = (p_A, p_B, p_C) = (.4, .5, .1)$. What are the mean, variance, and standard deviation of his portfolio returns? Applying the formulas above, we find that

$$\text{E}(R_p) = \sum_{i=1}^{n} p_i \mu_i = (.4 \times .10) + (.5 \times .20) + (.1 \times .30) = .04 + .10 + .03 = .17,$$

$$\begin{aligned} \text{Var}(R_p) &= \sum_{i=1}^{n} p_i^2 \sigma_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j>i} p_i p_j \sigma_{i,j} \\ &= ((.4)^2 \times .0064) + ((.5)^2 \times .0144) + ((.1)^2 \times .0225) \\ &\quad + 2((.4 \times .5 \times .0036) + (.4 \times .1 \times .0012) + (.5 \times .1 \times .0036)) \\ &= (.001024 + .0036 + .000225) + 2(.00072 + .000048 + .00018) \\ &= .006745, \end{aligned}$$

$$\text{SD}(R_p) = \sqrt{\text{Var}(R_p)} = .0821. \qquad ■$$

Efficient portfolios.

How should an investor choose which n asset portfolio to hold? As in the two-asset case, investors should always choose a portfolio in the **efficient set**: that is, a portfolio that maximizes expected return for some fixed level of risk. Since the calculations are more difficult in this case, we let the computer take care of them for us.

Excel calculation: *Portfolio analysis with three risky assets*

The `three_assets.xlsx` workbook (Figure 4.B.2) allows us to analyze portfolios created from three risky assets. Let's use this workbook to analyze the previous example.

Figure 4.B.2: three_assets.xlsx

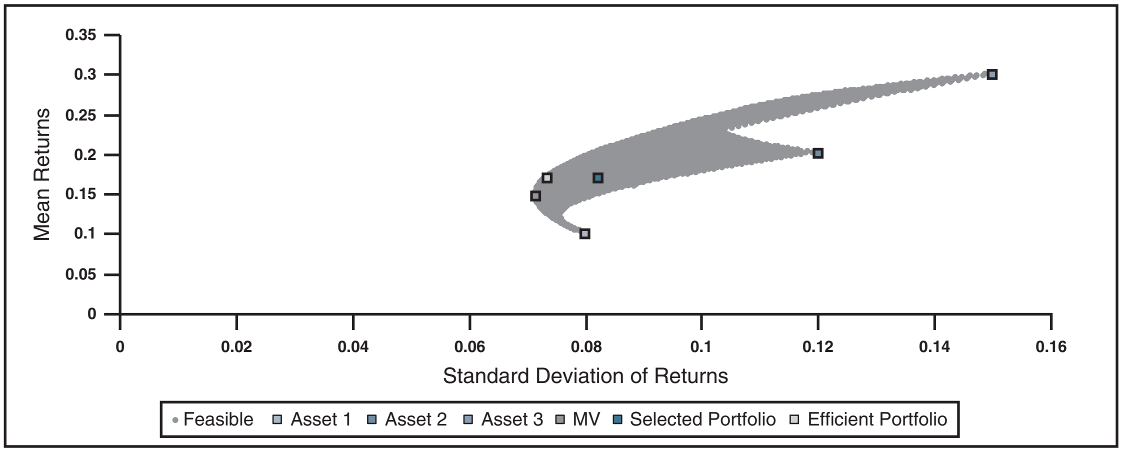
Three Risky Assets

Mean Returns	
Asset 1	0.1
Asset 2	0.2
Asset 3	0.3

Variance - Covariance Matrix		
0.0064	0.0036	0.0012
0.0036	0.0144	0.0036
0.0012	0.0036	0.0225

Standard Deviations	
SD(R_1)	0.0800
SD(R_2)	0.1200
SD(R_3)	0.1500

Correlations	
Corr(R_1,R_2)	0.3750
Corr(R_1,R_3)	0.1000
Corr(R_2,R_3)	0.2000



Minimum Variance Portfolio	
p_1^{mv}	0.6901
p_2^{mv}	0.1415
p_3^{mv}	0.1685
E(R^{mv})	0.1478
Var(R^{mv})	0.0051
SD(R^{mv})	0.0716

Efficient Portfolio	
Desired E(R_p)	0.17
p_1	0.5496
p_2	0.2008
p_3	0.2496
Var(R_p)	0.0054
SD(R_p)	0.0735

Selected Portfolio	
$\mathbf{p_1}$	0.400
$\mathbf{p_2}$	0.500
$\mathbf{p_3}$	0.100
E(R_p)	0.1700
Var(R_p)	0.0067
SD(R_p)	0.0821

Minimum p_i	0

To begin, type the mean vector and covariance matrix into the gray cells at the top of the worksheet. When you type the covariance matrix into the workbook, you only need to type in the entries on and above the diagonal: since $\sigma_{j,i} = \sigma_{i,j}$, the workbook fills in the rest automatically. The workbook presents the standard deviations σ_i and correlations $\rho_{i,j} = \sigma_{i,j}/(\sigma_i\sigma_j)$ as well.

Like the two-asset worksheet, the present one draws the feasible set of (σ, μ) points generated by portfolios of the three stocks. Some new features stick out. First, and most obviously, the feasible set is no longer just a curve, but an entire region in (σ, μ) space. Among other things, this means that if we pick an arbitrary portfolio, its (σ, μ) point is almost certain to lie in the interior of the feasible set, implying that the portfolio is not efficient.

For instance, enter the portfolio $(p_A, p_B, p_C) = (.4, .5, .1)$ from the previous example in the gray cells under "Selected portfolio." You need only enter the first two numbers; the third is computed automatically. The workbook computes the expected return and risk of this portfolio, $\text{E}(R_p) = .17$ and $\text{SD}(R_p) = .0821$, and then plots the point $(\text{SD}(R_p), \text{E}(R_p))$ in the diagram. As expected, this point is in the interior of the feasible set. This means that we would never actually want to choose the portfolio $p = (.4, .5, .1)$, since there are other portfolios that dominate it.

How do we find such portfolios? Suppose we would like a portfolio with a mean return of .17. The (σ, μ) diagram shows that there are an infinite number of points in the shaded region whose μ value is .17. Of all of these, there is one we like best: the one furthest to the left, whose risk is lowest. Finding the corresponding portfolio by hand requires a difficult calculation. Fortunately, the workbook can do this for us automatically. Look below the graph for the table called "Efficient portfolio," and type .17 into the gray cell labeled "Desired $\mathrm{E}(R_p)$." The workbook will calculate the lowest risk portfolio with this expected return and plot its (σ, μ) point in the diagram. In this case, the portfolio in question is $(p_A, p_B, p_C) = (.5496, .2008, .2496)$, whose returns have traits $\mathrm{E}(R_p) = .17$ (of course) and $\mathrm{SD}(R_p) = .0735$. This standard deviation is less than the standard deviation of .0821 generated by the (.4, .5, .1) portfolio, and as we can see, our new portfolio is represented by a point on the boundary of the feasible set.

Determining the minimum variance portfolio also involves a complicated calculation. The workbook takes care of this for us as well. In the present case, the workbook finds

$$\begin{aligned} p^{mv} &= (p_A^{mv}, p_B^{mv}, p_C^{mv}) = (.6901, .1415, .1685), \\ \mathrm{E}(R_{p^{mv}}) &= .1478, \\ \mathrm{SD}(R_{p^{mv}}) &= .0716. \end{aligned}$$

The minimum variance point $(\mathrm{SD}(R_{p^{mv}}), \mathrm{E}(R_{p^{mv}}))$ is plotted in (σ, μ) space too; by definition, it is the leftmost feasible point.

This example shows that while portfolio choice becomes a bit more complicated when there are many assets available for investment instead of just two, the basic principles remain the same. As before, the efficient frontier is the set of undominated (σ, μ) points: that is, the set of points on the northwest frontier of the feasible set. Also as before, the point on the efficient frontier that is furthest to the left is the minimum variance point $(\mathrm{SD}(R_{p^{mv}}), \mathrm{E}(R_{p^{mv}}))$.

The efficient frontier is contained within the boundary of the feasible set. Since nearly all portfolios generate returns in the interior of the feasible set, it follows that *nearly all portfolios are inefficient*. Thus, even without knowing an investor's risk preferences, we can substantially narrow down the set of portfolios he should consider.

■ Example *continued.*

In the previous example, Colic had the highest expected return ($\mu_C = .30$) and the highest risk ($\sigma_C = .15$). Is putting all of your capital in Colic the best way to earn a 30% expected return? To find out, return to the Excel worksheet and type .30 into the gray cell labeled "Desired E(R_p)." If the all-Colic portfolio were optimal, the worksheet would return the portfolio $(p_A, p_B, p_C) = (0, 0, 1)$. But it doesn't. Instead, it picks the portfolio $(p_A, p_B, p_C) = (-.2745, .5490, .7255)$, which achieves the expected return of $\mathrm{E}(R_p) = .30$ with a level of risk of only $\mathrm{SD}(R_p) = .1341$. Since p_A is negative, portfolio p involves short sales. By taking a negative position in Acme, we can invest more than 100% of our capital in Bake and Colic. This allows us to diversify our portfolio, lowering its risk, while maintaining a 30% expected return.[4]

When short selling is possible, it rarely makes sense to put all of your eggs in one basket. For each "pure" portfolio like $(p_A, p_B, p_C) = (0, 0, 1)$, we can typically find another with the same expected return but less risk. ■

4.B.2 Riskless lending and borrowing

To this point we've worked under the assumption that no investment is risk free. This is reflected in the fact that the minimum variance portfolio p^{mv} has always exhibited a strictly positive variance of returns. But capital invested in U.S. Treasury bills earns a return that is essentially without risk. How does including a riskless asset affect portfolio selection?

Mixed portfolios and their traits.

Suppose we are able to invest our capital not only in risky assets $1, \ldots, n$ with returns $R_1, \ldots, R_n$, but also in a riskless asset with rate of return r. We can represent mixed portfolios as lists with $1 + n$ components.

$$q = (q_0; q_1, \ldots, q_n).$$

As before, components q_1 through q_n represent the proportions of capital invested in each of the risky assets. The new component q_0 represents the proportion of capital invested in the riskless asset. By definition, these $1 + n$ portfolio weights must sum to one. The same logic used earlier shows that the returns on the mixed portfolio are

$$R_q = q_0 r + \sum_{i=1}^{n} q_i R_i.$$

It's often easiest to describe mixed portfolios in a somewhat different way: we specify the proportion s of our capital invested in some risky portfolio p, with the understanding that the remaining proportion is invested in the riskless asset.

[4] In the `three_assets.xlsx` workbook, the point $(\sigma, \mu) = (.1341, .30)$ corresponding to this efficient portfolio appears to the left of the shaded region. This is because the shaded region represents (σ, μ) points generated by portfolios without short sales. To make the shaded region allow for short sales, change the value of the gray cell labeled "Minimum p_i" at the lower left to a negative number.

In this case, the overall proportion of capital invested in risky asset i is $q_i = s\,p_i$, while the proportion invested in the riskless asset is $q_0 = 1 - s$. Thus the **mixed portfolio generated by *s* and *p*** is

$$q = (q_0; q_1, \ldots, q_n) = (1 - s; sp_1, \ldots, sp_n).$$

We can express the returns on this mixed portfolio q in terms of the riskless return and the returns on the risky portfolio p:

$$\begin{aligned} R_q &= q_0 r + \sum_{i=1}^{n} q_i R_i \\ &= (1 - s)r + \sum_{i=1}^{n} sp_i R_i \\ &= (1 - s)r + sR_p. \end{aligned}$$

What are the traits of this mixed portfolio? We find them using our formulas from Chapter 4, keeping in mind that the expected value of a constant is the constant itself, and that variances and covariances involving constants equal zero (see Exercise 4.2.4).

$$\begin{aligned} \mathrm{E}(R_q) &= \mathrm{E}((1 - s)r + sR_p) \\ &= \mathrm{E}((1 - s)r) + \mathrm{E}(sR_p) \\ &= (1 - s)r + s\mathrm{E}(R_p) \\ &= r + s(\mathrm{E}(R_p) - r), \\ \mathrm{Var}(R_q) &= \mathrm{Var}((1 - s)r + sR_p) \\ &= \mathrm{Var}((1 - s)r) + \mathrm{Var}(sR_p) + 2\,\mathrm{Cov}((1 - s)r, sR_p) \\ &= 0 + \mathrm{Var}(sR_p) + 0 \\ &= s^2\,\mathrm{Var}(R_p), \\ \mathrm{SD}(R_q) &= \sqrt{\mathrm{Var}(R_q)} \\ &= \sqrt{s^2\,\mathrm{Var}(R_p)} \\ &= s\,\mathrm{SD}(R_p) \qquad \text{if } s \geq 0. \end{aligned}$$

The key conclusion of this analysis is that $\mathrm{E}(R_q)$ and $\mathrm{SD}(R_q)$ are both *linear* functions of s, the proportion of capital invested in risky assets. For an interpretation, suppose that we change the proportion of capital invested in the risky portfolio from s to $s + t$. Then expected returns increase by $t \times (\mathrm{E}(R_p) - r)$, while the standard deviation of returns increases by $t \times \mathrm{SD}(R_p)$. Thus, regardless of the values of s and t, we have that

$$\frac{\text{Change in expected return}}{\text{Change in SD of return}} = \frac{t \times (\mathrm{E}(R_p) - r)}{t \times \mathrm{SD}(R_p)} = \frac{\mathrm{E}(R_p) - r}{SD(R_p)}.$$

In other words, shifting capital from the riskless asset to the risky portfolio p always alters expected return and risk in the fixed ratio above. This ratio is important enough to have a name.

Definition.

The **Sharpe ratio** for portfolio p, denoted SR_p, is defined by

$$SR_p = \frac{\mathrm{E}(R_p) - r}{\mathrm{SD}(R_p)}.$$

It describes the tradeoff between expected return and risk of moving capital from the riskless asset with return r to the risky portfolio p.

In Figure 4.B.3, we graph the feasible points in (σ, μ) space generated by mixtures of the riskless asset and the risky portfolio p. If we set $s = 0$, so that all capital is invested in the riskless asset, returns are represented by the point ZR (for “zero risk”) with coordinates $(0, r)$. If we set $s = 1$, so that all capital is invested in the risky portfolio p, we obtain the point P with coordinates $(\mathrm{SD}(R_p), \mathrm{E}(R_p))$. The fact that $\mathrm{E}(R_q)$ and $\mathrm{SD}(R_q)$ are linear functions of s means that as we increase s, the trait pairs $(\mathrm{SD}(R_q), \mathrm{E}(R_q))$ move in a straight line from ZR through P.

By definition, the slope of a straight line is the change in the vertical component (“the rise”) divided by the change in the horizontal component (“the run”) obtained by comparing any two points on the line. Using points ZR and P, we

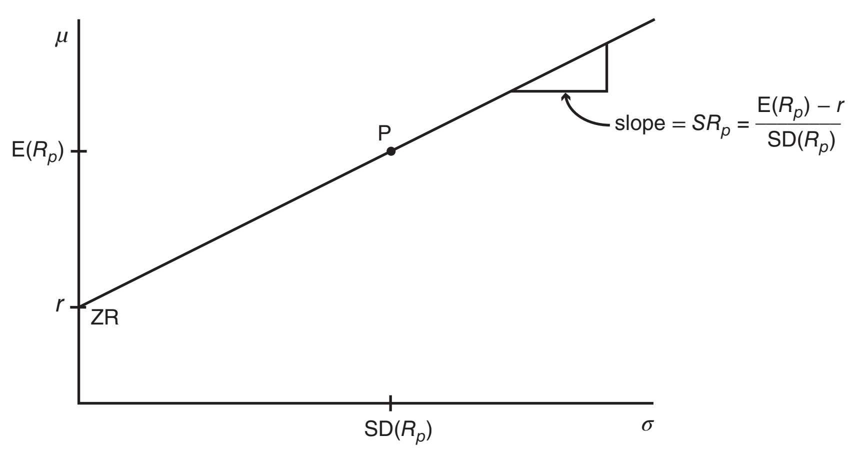


Figure 4.B.3: Riskless lending and borrowing.

obtain a rise of $E(R_p) - r$ and a run of $SD(R_p) - 0 = SD(R_p)$, for a ratio of $(E(R_p) - r)/\, SD(R_p)$. In other words,

Graphing the traits of mixed portfolios.
The line describing the traits of portfolios that mix the riskless asset with return r and the risky portfolio p has an intercept at $(0, r)$ and a slope equal to the Sharpe ratio $SR_p = (E(R_p) - r)/\, SD(R_p)$.

Points on the line to the right of P represent mixed portfolios in which $s > 1$: more than 100% of capital is invested in the risky asset, and a negative amount of capital is invested in the riskless asset (since $1 - s < 0$). Taking a negative position in the riskless asset means borrowing money at interest rate r. Doing so generates high expected returns, but at the cost of high levels of risk.[5]

Example Suppose the portfolio p has an expected return of $E(R_p) = .16$ and a standard deviation of returns of $SD(R_p) = .10$. If the riskless asset has a rate of return of .04, then the Sharpe ratio of portfolio p is

$$SR_p = \frac{E(R_p) - r}{SD(R_p)} = \frac{.16 - .04}{.10} = 1.2.$$

A cautious investor places one-quarter of his capital in portfolio p and the other three-quarters in the riskless asset. The mean return and standard deviation of returns from this mixed portfolio are

$$E(R_q) = r + s(E(R_p) - r) = .04 + .25(.16 - .04) = .07;$$
$$SD(R_q) = s\, SD(R_p) = .25 \times .10 = .025.$$

A very aggressive investor puts double her capital in the portfolio p, borrowing the full value of her capital at the riskless rate in order to do so. The traits of her mixed portfolio are

$$E(R_q) = r + s(E(R_p) - r) = .04 + 2(.16 - .04) = .28;$$
$$SD(R_q) = s\, SD(R_p) = 2 \times .10 = .20.$$

The expected return of this mixed portfolio is much higher than that of the cautious investor, but so is the level of risk.

[5] In reality, most of us must borrow money at an interest rate that is higher than the rate of return we can obtain by lending (i.e., the rate of return on U.S. Treasury bills). But for large investors, borrowing and lending rates are nearly equal. In any case, our analysis can be extended to include different rates of return for borrowing and lending.

The capital market line.

We are now ready to consider the general portfolio selection problem. Suppose we can invest in n risky assets with returns $R_1, \dots, R_n$ and a riskless asset with return r. Make the reasonable assumption that the riskless return r is less than the expected return $\mathrm{E}(R_{p^{mv}})$ of the minimum variance portfolio. Which points in (σ, μ) space correspond to efficient mixed portfolios?

Let's collect the three observations we need to answer this question.

Observation 1: When only risky assets are available for investment, the feasible set takes the form of the shaded region in Figure 4.B.4(i).

Observation 2: If we fix a portfolio p of risky assets, the returns on mixed portfolios based on p are represented by points on the straight line from the point ZR, representing the riskless return, through the point P, representing the return on portfolio p, as drawn in Figure 4.B.4(ii).

Observation 3: Efficient portfolios are represented by undominated (σ, μ) points: that is, (σ, μ) points that are as far northwest as possible.

How do these three observations enable us to determine the efficient mixed portfolios?

Intuitively, we want to split our investment between the riskless asset and a risky portfolio p, choosing p in such a way that the line from Figure 4.B.4(i) will lie as far northwest as possible. How do we find that line? Simple: Place your pencil on Figure 4.B.4(i), putting the eraser at the zero-risk point ZR and pointing the tip straight up. Then, holding the eraser in place, pivot the pencil clockwise until it just hits the shaded region (Figure 4.B.4(iii)). The point of contact, labeled T, represents the returns on the risky portfolio we seek.

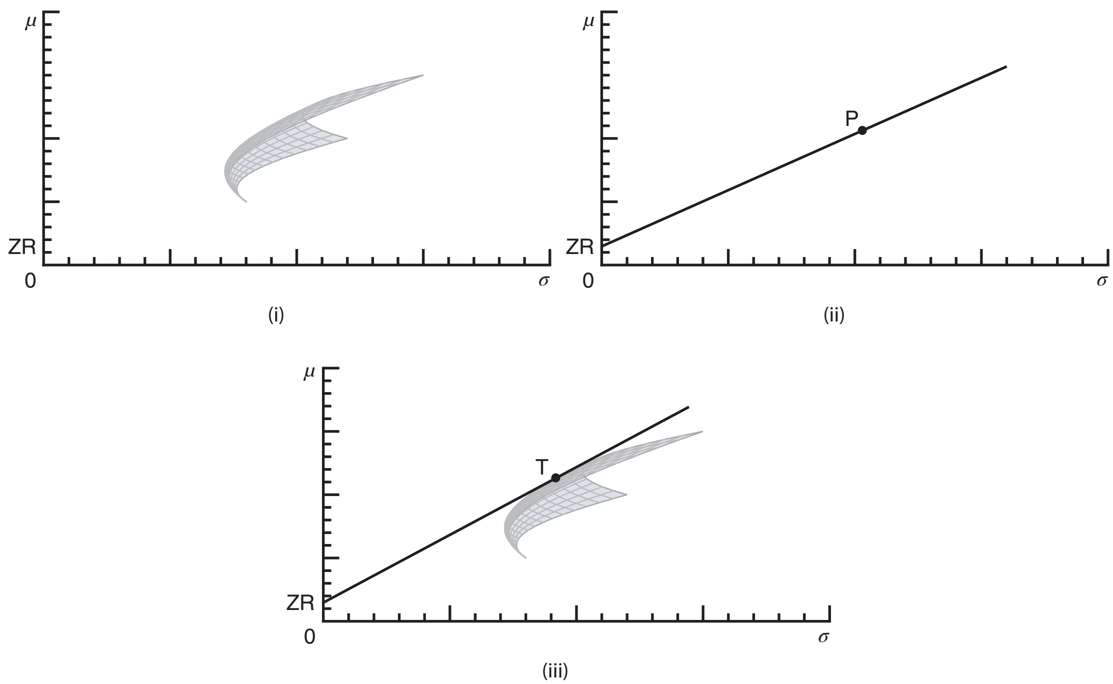
To ease the discussion to follow, let's introduce a few definitions.

Definitions.

The straight line emanating from the zero-risk point ZR and resting on the edge of the feasible set is called the **capital market line**. The point of contact between the capital market line and the feasible set, labeled T, is called the **tangent point**, and the portfolio p^t that generates this point is called the **tangent portfolio**.

The points on the capital market line are precisely those (σ, μ) points that are efficient when both risky and riskless assets are available for investment. The points on this line are clearly feasible: they are generated by mixed portfolios that combine the riskless asset and the tangent portfolio p^t. These points are also undominated: were there a feasible point to the northwest of the capital market line, we would have stopped rotating our pencil sooner than we actually did.

Figure 4.B.4: Constructing the capital market line



The capital market line can be described in a different but equivalent way: it is the line that starts at the zero-risk point ZR, passes through some point in the shaded region, and has the greatest possible slope.[6] But as we saw in Section 4.B.2, the slope of the line from the point ZR through the point $(\mathrm{SD}(R_p), \mathrm{E}(R_p))$ is portfolio p's Sharpe ratio, $SR_p = (\mathrm{E}(R_p) - r)/\,\mathrm{SD}(R_p)$. Combining these facts yields the following conclusion.

The tangent portfolio, the Sharpe ratio, and the capital market line.
The tangent portfolio p^t is the portfolio of risky assets with the highest Sharpe ratio. This maximal Sharpe ratio, SR_{p^t}, is the slope of the capital market line.

The ideas behind the tangent portfolio and the capital market line are easy to show in pictures. On the other hand, if we start with the numerical data describing asset returns, the pictures don't tell us how to actually compute the

[6] In other words, we can also find the capital market line by laying our pencil on Figure 4.B.4(i) *horizontally* and rotating it *counterclockwise* until the last moment of contact with the feasible set.

tangent portfolio. Since this portfolio maximizes the Sharpe ratio, it shouldn't surprise you to learn that the formula for the tangent portfolio is obtained by solving a calculus problem. In examples with just two risky assets, the calculus problem is long, but not too difficult (Exercise 4.B.17). In the end, we find that the weight on asset 1 in the tangent portfolio is

$$p_1^t = \frac{\sigma_{12}(\mu_2 - r) - \sigma_2^2(\mu_1 - r)}{\sigma_{12}(\mu_1 + \mu_2 - 2r) - \sigma_1^2(\mu_2 - r) - \sigma_2^2(\mu_1 - r)}.$$

Since there are two risky assets, the weight on asset 2 is $p_2^t = 1 - p_1^t$.

When there are three or more risky assets, the formula for p^t is much more complicated. But since we understand the ideas behind this formula, we won't worry about the details; instead, we let the computer take care of them for us.

Excel calculation: *Portfolio selection with three risky assets and a riskless asset*

Let's return once more to the Acme, Bake, and Colic example from the last section. We saw that when our portfolios only used these three risky assets, the best way to obtain a mean return of $E(R_p) = .17$ was to invest in the portfolio $(p_A, p_B, p_C) = (.5496, .2008, .2496)$, whose returns have standard deviation $SD(R_p) = .0735$. Now suppose that there is also a riskless asset with a rate of return of .05. Does using this asset allow us to obtain a mean return of .17 with less exposure to risk?

From the previous discussion, we expect the efficient portfolio with expected return .17 to combine the tangent portfolio and the riskless asset. To find this efficient portfolio, open the workbook called `three_plus_one.xlsx` (Figure 4.B.5) and type the traits of the risky returns and the riskless rate of returns into the appropriate gray cells. The workbook returns the tangent portfolio p^t, which in this case is $(p_A^t, p_B^t, p_C^t) = (.0911, .3946, .5143)$. The expected return and risk of this portfolio are $E(R_{p^t}) = .2423$ and $SD(R_{p^t}) = .1004$. Since the expected return is higher than .17, the efficient portfolio we seek will combine this portfolio with the riskless asset.

To find this mixed portfolio, type .17 into the gray cell labeled "Desired $E(R_p)$." When you do so, the workbook returns a portfolio with total weight $s = .6239$ on risky assets, and hence weight $q_0 = 1 - s = .3761$ on the riskless asset. The weights on the individual risky assets are $(q_A, q_B, q_C) = (sp_A^t, sp_B^t, sp_C^t) = (.0568, .2462, .3209)$. The workbook also reports the standard deviation of this portfolio's return, $SD(R_q) = .0626$, and it plots this portfolio's (σ, μ) point on the capital market line.

Figure 4.B.5: three_plus_one.xlsx

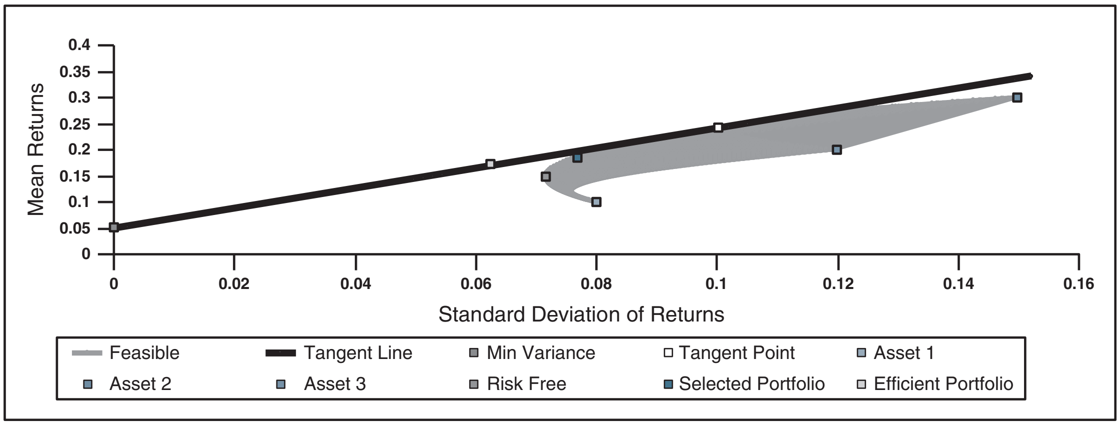
Three Plus One

Mean Returns	
Asset 1	0.1
Asset 2	0.2
Asset 3	0.3
Risk Free Rate	0.05

Variance - Covariance Matrix		
0.0064	0.0036	0.0012
0.0036	0.0144	0.0036
0.0012	0.0036	0.0225

Standard Deviations	
$SD(R_1)$	0.0800
$SD(R_2)$	0.1200
$SD(R_3)$	0.1500

Correlations	
$Corr(R_1,R_2)$	0.375
$Corr(R_1,R_3)$	0.100
$Corr(R_2,R_3)$	0.200



Minimum Variance Portfolio	
p_1^{mv}	0.69006
p_2^{mv}	0.14147
p_3^{mv}	0.16847
$E(R^{mv})$	0.1478
$Var(R^{mv})$	0.0051
$SD(R^{mv})$	0.0716

Tangent Portfolio	
p_1^t	0.0911
p_2^t	0.3946
p_3^t	0.5143
$E(R_p^t)$	0.2423
$Var(R_p^t)$	0.0101
$SD(R_p^t)$	0.1004
Sharpe Ratio	1.9156

Efficient Portfolio	
Desired $E(R_p)$	0.17
p_1	.0911
p_2	.3946
p_3	.5143
s	.6239
$SD(R_p)$	.0626
Sharpe Ratio	1.9156

Selected Portfolio	
p_1	0.1
p_2	0.65
p_3	0.25
s	0.8
$E(R_p)$	0.1820
$SD(R_p)$	0.0770
Sharpe Ratio	1.7154

Minimum p_i	0

Excel calculation: *Finding the highest Sharpe ratio**

We saw earlier that the tangent portfolio p^t is the risky portfolio with the largest Sharpe ratio. Let's use this fact to find the tangent portfolio directly using Excel's maximization functions.

Open the workbook called `Sharpe_ratio.xlsx`, and enter the risky assets' traits and the riskless rate of return. If you type in a list of risky portfolio weights, the worksheet returns the portfolio's Sharpe ratio. It also draws a line from the zero-risk point through the point representing your risky portfolio.

If you try lots of different portfolios, then you can eventually find the tangent portfolio by trial and error. But why not let Excel do the work for you? Selecting "`Solver`" in the "`Tools`" menu opens a window called "`Solver Parameters`". Since you want to maximize the Sharpe ratio, which lives in cell `F37`, type "`F37`" in the box called "`Set Target Cell`". You want to maximize this ratio through your choice of portfolio weights, which live in cells `F31` through `F33`. To do so, enter "`F31:F32`" in the box labeled "`By Changing Cells`". (Cell `F33`, representing the weight on the third risky asset, is determined by the weights on the first two risky assets.) Once you've done all this, click on "`Solve`". Before your very eyes, the tangent portfolio appears in cells `F31` through `F33`, the highest Sharpe ratio appears in cell `F37`, and the capital market line is drawn in the accompanying picture.

The separation principle.

In the arguments from the previous section, we emphasized the geometry that underlies the capital market line. It's worthwhile to recast this analysis from other points of view.

Let's focus on the form of efficient mixed portfolios. From this perspective, our analysis says that every efficient mixed portfolio is a combination of the risk free asset and the tangent portfolio p^t. Thus, while risk preferences determine how much capital to put in the riskless asset and how much to put in risky assets, whatever capital is invested in risky assets should be invested in the proportions described by the vector p^t.

Efficient mixed portfolios.

Every efficient mixed portfolio holds risky assets in identical proportions: namely, the proportions described by the tangent portfolio p^t.

At first glance, this result may seem unintuitive: it is natural to expect that a change in your risk preferences would lead you to change your relative shares of capital in different risky assets. For instance, if you became more risk tolerant, you might expect it to be optimal to shift funds from low-risk, low-return stocks to high-risk, high-return stocks. This intuition is false. Instead, you should shift some of your funds from the riskless asset to risky assets without altering your relative holdings of risky assets.

We can recast this property as a procedure for selecting mixed portfolios.

The separation principle.

The choice of an optimal mixed portfolio can be made in two steps. First, determine the tangent portfolio p^t*. Second, decide what proportion of capital to invest in* p^t *and what proportion to invest in the riskless asset.*

The key insight here is that the ideal portfolio of risky assets does not depend on risk preferences. Whether we are very risk tolerant or very risk averse, the first step in our analysis is to compute the tangent portfolio p^t. Only after this do we let our risk preferences enter our decision, allowing them to determine how we split our capital between the portfolio p^t and the riskless asset.

4.B.3 The market portfolio

We obtain the most profound consequences of the separation principle when we use it to study the choices made by all investors in the market at once. To do this, we need to describe all investors' beliefs about assets' future performances. The simplest assumption we can make is that their beliefs coincide.

Assumption: Homogeneous expectations

All investors share the same beliefs about assets' returns. In particular, all investors agree about the values of the return means $E(R_i) = \mu_i$, the return variances $Var(R_i) = \sigma_i^2$, and the return covariances $Cov(R_i, R_j) = \sigma_{i,j}$.

Of course, real investors differ in their beliefs about how different assets will perform: one investor may think that Microsoft will have an extraordinarily good year, while others may predict that Microsoft's performance will be subpar. But the point of the model is not to create an exact depiction of reality. Even if we could write down such a model, it would be far too complicated to be of any practical use. Instead, we want to create a model that trades off realism and simplicity in a way that gives us valuable insights about how financial markets work.

What are the consequences of homogeneous expectations? Under this assumption, each investor computes the same tangent portfolio p^t, and then lets his risk preferences determine how to split his capital between the portfolio p^t and the riskless asset. We have not assumed that all investors share the same risk preferences: conservative investors will put most of their capital in the riskless asset, while aggressive investors will put 100% or more of their capital into risky assets. But however much capital an investor places in the risky assets, he holds these assets in the proportions p^t. Therefore, if we add up risky asset holdings over all investors, we find that the market's aggregate risky holdings are described by the portfolio p^t as well.

■ Example To reinforce this point, let's look at a simple example. Suppose there are three assets available for investment and that the tangent portfolio is $p^t = (p_1^t, p_2^t, p_3^t) = (.5, .4, .1)$. Then every investor should hold risky assets in the ratio 5 to 4 to 1.

Now suppose that the participants in this market are named Alicia, Bennie, Carlo, and Dave. These investors have different amounts of capital and different risk preferences, but by the separation principle, they hold risky assets in the proportions stated above.

				asset holdings ($)			
investor	capital ($)	risk tolerance	s	riskless	asset 1	asset 2	asset 3
Alicia	200,000	Very high	1.5	−100,000	150,000	120,000	30,000
Bennie	400,000	High	1.0	0	200,000	160,000	40,000
Carlo	250,000	Moderate	.8	50,000	100,000	80,000	20,000
Dave	500,000	Low	.2	400,000	50,000	40,000	10,000
				totals:	500,000	400,000	100,000

Focus on the last three columns of the table. Despite their differing amounts of capital and levels of risk tolerance, each of our four investors holds risky assets in proportions 5 to 4 to 1. Therefore, when we add their dollar holdings up, we find that the aggregate holdings in the market are also in proportions 5 to 4 to 1. This is what we mean when we say that the market's aggregate risky holdings are described by the portfolio p^t. ■

In summary: Under the assumption of homogeneous expectations, each investor holds risky assets in the proportions given by the tangent portfolio p^t. Therefore, if we sum risky asset holdings over all investors, aggregate holdings in risky assets also follow the proportions p^t. When we want to emphasize this conclusion, we refer to the tangent portfolio as the **market portfolio**, and we use the notation p^m in place of p^t.

The fact that the market portfolio is the tangent portfolio is the linchpin of our analysis of the relationship between risk and expected return.

4.B.4 Risk and expected returns

Two kinds of risk.

In Chapter 3 we introduced the variance as a measure of a random variable's dispersion. A primary motivation for this concept was that in financial contexts, variance provides a measure of an investment's riskiness.

Now it's time to be more precise about what sort of risk variance is measuring. By definition, the variance of the returns on risky portfolio p is the average squared deviation of the portfolio's return from its expected return:

$$\text{Var}(R_p) = \sum_x (x - \text{E}(R_p))^2\, \text{P}(R_p = x).$$

If we are thinking about holding portfolio p, this measure of dispersion captures risk of the sort that matters to us.

But there is another, distinct sort of risk that is not captured by variances. Suppose we are holding a diverse portfolio p, and that we want to slightly increase our stake in asset i. In order to do this, we sell fraction ε of our original portfolio and use the proceeds to buy more of asset i. If we let p^i denote the portfolio that holds only asset i, and if we let p^ε denote our new portfolio, then[7]

$$p^\varepsilon = \varepsilon p^i + (1-\varepsilon)p.$$

The return on this new portfolio is

$$R_{p^\varepsilon} = \varepsilon R_i + (1-\varepsilon)R_p.$$

How does the variance of this portfolio's returns compare to the variance of the returns of our original portfolio p? To find out, let's first compute the variance of R_{p^ε} using our trait formulas:

$$\begin{aligned}
\mathrm{Var}(R_{p^\varepsilon}) &= \mathrm{Var}(\varepsilon R_i + (1-\varepsilon)R_p) \\
&= \mathrm{Var}(\varepsilon R_i) + \mathrm{Var}((1-\varepsilon)R_p) + 2\,\mathrm{Cov}(\varepsilon R_i, (1-\varepsilon)R_p) \\
&= \varepsilon^2\,\mathrm{Var}(R_i) + (1-\varepsilon)^2\,\mathrm{Var}(R_p) + 2\varepsilon(1-\varepsilon)\,\mathrm{Cov}(R_i, R_p) \\
&= \varepsilon^2\,\mathrm{Var}(R_i) + (1-2\varepsilon+\varepsilon^2)\,\mathrm{Var}(R_p) + (2\varepsilon-2\varepsilon^2)\,\mathrm{Cov}(R_i, R_p)
\end{aligned}$$

To compare this variance to our original return variance, we compute the difference.

$$\mathrm{Var}(R_{p^\varepsilon}) - \mathrm{Var}(R_p) = \varepsilon^2\,\mathrm{Var}(R_i) + (-2\varepsilon+\varepsilon^2)\,\mathrm{Var}(R_p) + (2\varepsilon-2\varepsilon^2)\,\mathrm{Cov}(R_i, R_p).$$

Now if ε is small, then ε^2 is extremely small, so we can conclude that

$$\mathrm{Var}(R_{p^\varepsilon}) - \mathrm{Var}(R_p) \approx 2\varepsilon(\mathrm{Cov}(R_i, R_p) - \mathrm{Var}(R_p)).$$

We can express our findings as follows:

The effects of small trades on the variance of portfolio returns.
If we sell fraction ε of portfolio p and use the proceeds to buy more of asset i, the difference between the variances of returns on our new and old portfolios is approximately $2\varepsilon(Cov(R_i, R_p) - Var(R_p))$.

We began this section by restating an old fact: the risk of holding a portfolio p is measured by the variance of R_p. But now we are considering a new sort of risk: the *change in variance* that is generated when we substitute a small amount of asset i to a diverse portfolio. The argument above shows that this variety of risk is measured by the *covariance* between the returns of asset i and the returns of portfolio p.

It's helpful to introduce some new terms to distinguish between the two types of risk considered above. The risk associated with adding a small amount of a particular asset to a diverse portfolio—the sort of risk measured by covariances—we

[7]More explicitly, $p^\varepsilon_j = (1-\varepsilon)p_j$ when $j \neq i$, and $p^\varepsilon_i = \varepsilon + (1-\varepsilon)p_i$.

call **marginal risk**, since this is the risk that is relevant when we make marginal changes in our portfolio. We can also think of marginal risk as capturing an asset's suitability as a **hedge**: in other words, its ability to lower the variance of returns of the reference portfolio. By contrast, the risk of placing all of one's capital in a particular portfolio—that is, the sort of risk measured by variances—we call **absolute risk**.

We now have proposed our two types of risk. In the remainder of this appendix, we will argue that only the bearing of *marginal risk* is associated with high expected returns.

Example If we're measuring the average height of people in a classroom and a very tall person walks through the door, then the average height increases. Does the same thing happen with absolute risk when we add to our portfolio a new stock whose absolute risk is higher than that of our current collection of stocks?

Not necessarily. Imagine a stock with very high absolute risk: its return variance is twice as high as that of our current portfolio. When we add this new stock to our portfolio, we might expect it to increase the portfolio's absolute risk level. But suppose that the return on the new stock is highly negatively correlated with the return on our original portfolio. Then, when our portfolio increases in value, the new stock tends to perform poorly, but when our portfolio slumps, the new stock tends to perform well. Under these circumstances, adding a little of the new "absolutely risky" asset actually *decreases* the absolute risk of our portfolio. In fact, the new asset can even reduce absolute risk if the correlation between its returns and our portfolio's returns is positive, as long as it is not too large. ■

Beta.

Before showing that marginal risk is associated with high expected returns, we introduce a new way of measuring it. As we have seen, the marginal risk of an asset can be described by the covariance of its return with the return of a reference portfolio of risky assets. We know from Section 4.B.3 that each investor's holdings of risky assets are described by the same portfolio, the market portfolio p^m. We should therefore use p^m as the reference portfolio. If we do so and then rescale the result, we obtain our new measure of an asset's marginal risk.

Definition.

The **beta** of asset i, denoted β_i, is a measure of the marginal risk associated with asset i. It is defined as follows:

$$\beta_i = \frac{\mathrm{Cov}(R_i, R_{p^m})}{\mathrm{Var}(R_{p^m})}.$$

To obtain the beta of asset i, we find the covariance of asset i's returns with the returns on the market portfolio p^m, and then divide the result by the variance of the returns on p^m. To see why the rescaling is helpful, let's revisit our calculation from

Section 4.B.4. Replacing the reference portfolio p with the market portfolio p^m, we find that the difference between the variances of returns on the new portfolio $p^\varepsilon = \varepsilon p^i + (1 - \varepsilon)p^m$ and the market portfolio p^m is

$$\text{Var}(R_{p^\varepsilon}) - \text{Var}(R_{p^m}) \approx 2\varepsilon(\text{Cov}(R_i, R_{p^m}) - \text{Var}(R_{p^m})).$$

If we multiply and divide the right-hand side of this expression by $\text{Var}(R_{p^m})$ and then apply the definition of beta, we obtain

$$\text{Var}(R_{p^\varepsilon}) - \text{Var}(R_{p^m}) \approx 2\varepsilon\, \text{Var}(R_{p^m})(\beta_i - 1).$$

What does this last equation tell us?

Betas and the effects of small trades on absolute risk.

Adding a small amount of asset i to the market portfolio reduces absolute risk if and only if $\beta_i < 1$.

The value of β_i describes how good a hedge asset i provides against the absolute risk in the market portfolio p^m. A beta of 1 reflects an asset that is neutral as a hedge: adding a little more of this asset to the market portfolio neither increases nor decreases absolute risk. For its part, a beta of 0 means that an asset is as good a hedge as the riskless asset (see Exercise 4.B.7). It follows that if the beta of asset i is negative (or equivalently, if the covariance of asset i's return with the market return is negative), then asset i has a lower marginal risk than an asset whose return is a sure thing.

We summarize the interpretations of various ranges of values for beta in the table below.

value of β_i	effect on portfolio p^m of increasing the holdings of asset i
$\beta_i > 1$	adds risk to the overall portfolio
$\beta_i = 1$	does not affect risk
$1 > \beta_i > 0$	reduces risk, although not as well as the riskless asset
$\beta_i = 0$	reduces risk as well as the riskless asset
$\beta_i < 0$	reduces risk better than the riskless asset

WHICH KINDS OF STOCKS HAVE WHICH KINDS OF BETAS?

The definition of beta tells us which sorts of stocks will have high, moderate, low, and negative betas. An asset's beta is proportional to its covariance with the market portfolio, so any statements we make about betas are ultimately statements about this covariance. Since the market portfolio describes overall asset holdings in the economy, an asset's beta describes the extent to which the asset's returns tend to move in synchrony with returns in the economy as a whole.[8]

(*continued*)

[8]When computing betas, financial professionals use overall market indices—for instance, the Standard & Poor's 500 index or the Wilshire 5000 index—as proxies for the market portfolio.

(*continued*)

To illustrate this point, we report the November 2004 betas of stocks in a variety of firms in the table below.[9]

beta above 1		beta near 1		beta near 0		negative beta	
company	beta	company	beta	company	beta	company	beta
XM	3.51	GM	1.14	Merck	.21	Hewitt	−.11
Yahoo	3.40	GE	1.06	Coca-Cola	.20	Northrop	−.15
Amazon	2.39	Whirlpool	.96	General Mills	−.02	ADG	−.62

Starting at the left are firms with high betas, which tend to thrive when times are good and flounder when times are bad. An Internet firm like Yahoo that earns revenues from online advertising does very well when the economy is strong and very poorly when it is weak. Such firms have betas much greater than one. Companies with speculative business plans have high betas as well. A good example is XM Satellite Radio, whose future will be spectacular if its subscription-based radio service becomes popular and nonexistent if it does not.

The stocks of firms whose performances parallel but do not exacerbate the business cycle tend to have betas near one. Take firms like General Motors and Whirlpool: people tend to buy new cars and appliances when they have more money to spend, so the fortunes of companies that sell these products tend to parallel the business cycle. Large, diversified conglomerates like General Electric, which offer a wide variety of products and services, also tend to have betas near one.

Companies whose products are purchased steadily throughout the business cycle often have betas near zero. Whether the market is soaring or plummeting, people still eat cereal, drink soda, and take their medicine. Firms like General Mills and Coca-Cola that make consumer staples and firms like Merck that produce pharmaceuticals have sales figures that are fairly independent of economic conditions; these firms all have betas near zero.

What about negative betas? In November 2004, one commonly used stock screener listed 723 stocks with betas less than zero.[10] While many such companies are small and not actively traded, others are simply great hedges. One example is Hewitt Associates, a firm that specializes in human resources consulting. In economic downturns, many firms look to cut costs by laying off employees. Hewitt helps firms make these tough choices, so their business picks up when the economy is sluggish. Another group of stocks that often have negative betas are those of defense contractors. While the market often drops in times of war and uncertainty, these circumstances are positive boons for firms like Northrop Grumman and Allied Defense Group (ADG), suppliers of materiel to the Department of Defense. Holding the stocks of these companies reduces absolute risk even better than holding a Treasury bill whose return is certain.

[9] Data from Reuters via finance.yahoo.com.
[10] See www.msn.com/en-us/money.

The security market equation and the security market line.

Finally, we come to the climax of this appendix. What is the relationship between risk and expected return? We gave away part of the answer to this question already: of the two types of risk we defined above, it is marginal risk that is intimately connected with expected return. But what exactly is the connection? This link is described by a strikingly simple formula.

The security market equation. $E(R_i) - r = \beta_i(E(R_{p^m}) - r)$.

What does the security market equation say? The expression on the left, $\mathrm{E}(R_i) - r$, is the difference between the expected return on asset i and the riskless rate of return. This quantity is called the **risk premium** of asset i, or the **expected excess return** of asset i. Similarly, the expression in parentheses, $\mathrm{E}(R_{p^m}) - r$, is called the **market risk premium**. Using these terms, we can state the security market equation in words:

The risk premium of asset i equals the beta of asset i times the market risk premium.

One derivation of this equation is offered in Exercise 4.B.18.

We can use the security market equation to draw a graph that relates assets' betas to their expected returns. To do so, we rewrite the security market equation by adding r to each side.

$$\mathrm{E}(R_i) = r + \beta_i(\mathrm{E}(R_{p^m}) - r).$$

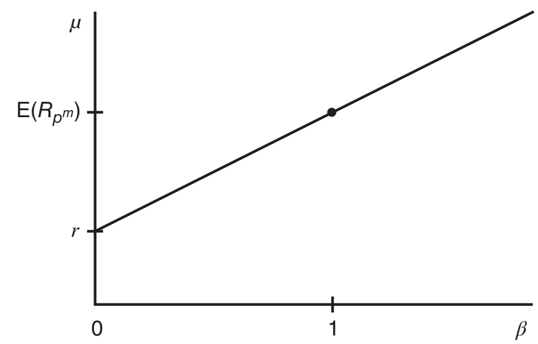
The graph of this equation, which forms a straight line in (β, μ) space (*not* (σ, μ) space), is called the **security market line** (Figure 4.B.6).

Example Suppose that the risk-free rate of return is $r = .05$, and that the expected market return is $\mathrm{E}(R_{p^m}) = .12$. Given the betas of stock in Yahoo, GE, Coca-Cola, and ADG from the previous table, then the expected returns on these stocks are

$$\mathrm{E}(R_Y) = r + \beta_Y(\mathrm{E}(R_{p^m}) - r) = .05 + 3.40 \times (.12 - .05) = .2880;$$
$$\mathrm{E}(R_{GE}) = r + \beta_{GE}(\mathrm{E}(R_{p^m}) - r) = .05 + 1.06 \times (.12 - .05) = .1242;$$
$$\mathrm{E}(R_{CC}) = r + \beta_{CC}(\mathrm{E}(R_{p^m}) - r) = .05 + .20 \times (.12 - .05) = .0640;$$
$$\mathrm{E}(R_{ADG}) = r + \beta_{ADG}(\mathrm{E}(R_{p^m}) - r) = .05 + (-.62) \times (.12 - .05) = .0066.$$

Our goal here was to understand the relationship between risk and expected return. With the security market equation in hand, we have achieved it. This equation tells us that there is a linear relationship between an asset's beta and its expected

Figure 4.B.6: The security market line.



return. Thus, an investor who holds assets with high levels of marginal risk is rewarded with high expected returns, while an investor who holds assets with significant absolute risk need not be rewarded at all.

Why should this be so? Ultimately, each investor cares about the dispersion of the returns of his full portfolio, not the dispersions of the returns of the individual assets therein. He is therefore willing to hold assets with high-variance returns if these assets serve as hedges against the risks contained in the rest of his portfolio.

Under the assumption of homogeneous expectations, every investor holds the same portfolio of p^m risky assets. If an asset has a low or negative beta, all investors will find the asset a good hedge, so demand for the asset will only be stable if its expected return is low. Similar logic applies to high-beta assets: since such assets have poor hedging properties, investors are only willing to hold them if they are compensated with high expected returns.

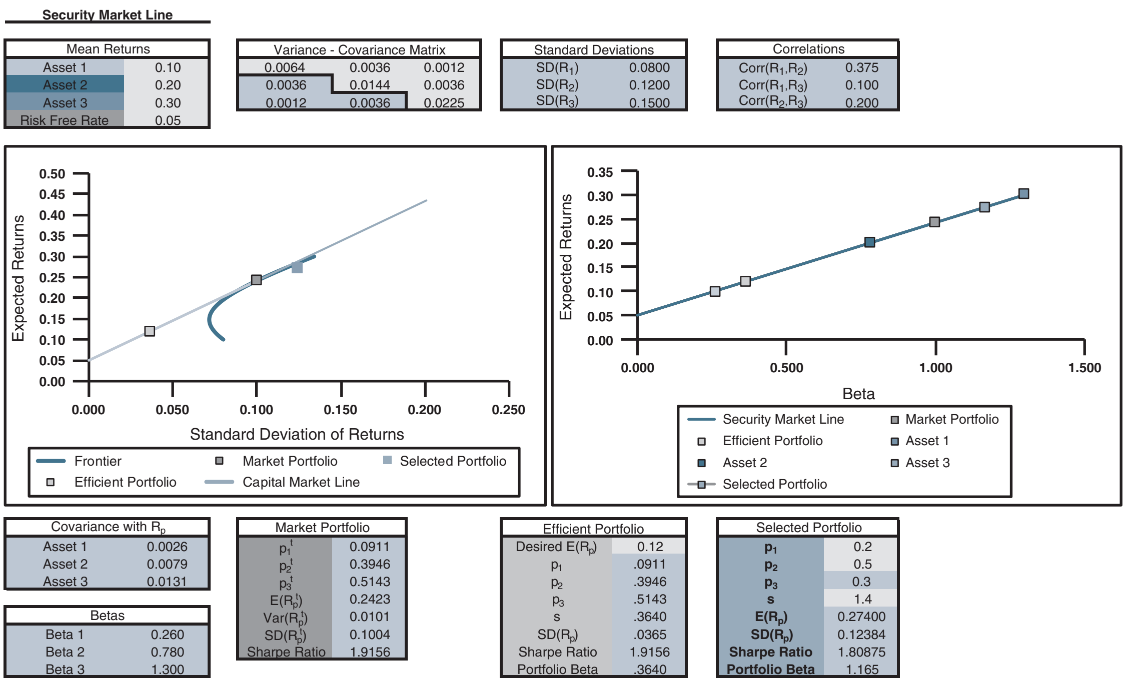
Our conclusions about the connections between betas, marginal risk, and expected returns are summarized in the following table.

value of β_i	marginal risk of asset i	expected return of asset i
$\beta_i > 1$	greater than that of p^m	greater than $\mathrm{E}(R_{p^m})$
$\beta_i = 1$	equal to that of p^m	equal to $\mathrm{E}(R_{p^m})$
$1 > \beta_i > 0$	between that of p^m and that of the riskless asset	between $\mathrm{E}(R_{p^m})$ and r
$\beta_i = 0$	equal to that of the riskless asset	equal to r
$\beta_i < 0$	less than that of the riskless asset	less than r

Excel calculation: *Betas and the security market line*

Let's return one last time to the Acme, Bake, and Colic example. Starting with a description of the traits of the assets' returns, the `three_plus_one.xlsx` workbook (Figure 4.B.5) computed the market portfolio $p^m = (p^m_A, p^m_B, p^m_C) = (.0911, .3946, .5143)$. It also reported the expected return on this portfolio, $\mathrm{E}(R_{p^m}) = .2423$, as well as the variance of its returns, $\mathrm{Var}(R_{p^m}) = .0101$.

Figure 4.B.7: security_market_line.xlsx



Let us complete our analysis using the `security_market_line.xlsx` workbook (Figure 4.B.7). This workbook computes the covariance between each asset's returns and the returns on the market portfolio; it then divides these numbers by $\mathrm{Var}(R_{p^m})$ to find the assets' betas:

$$\beta_A = .26, \quad \beta_B = .78, \text{ and } \beta_C = 1.30.$$

According to the security market equation, it is possible to express each asset's expected return, which was part of our initial description of the asset structure, in terms of the riskless return, the expected market return, and the asset's beta: $\mathrm{E}(R_i) = r + \beta_i(\mathrm{E}(R_{p^m}) - r)$. In the present example, ignoring roundoff errors, we find that

$$\mathrm{E}(R_A) = .05 + .26 \times (.2423 - 0.05) = .10,$$
$$\mathrm{E}(R_B) = .05 + .78 \times (.2423 - 0.05) = .20,$$
$$\mathrm{E}(R_C) = .05 + 1.30 \times (.2423 - 0.05) = .30.$$

These are indeed the mean returns with which we began the example.

In addition to drawing the feasible set and capital market line in (σ, μ) space, the `security_market_line.xlsx` workbook draws the security market line in (β, μ) space, and it plots points in this space representing the returns of each asset. Of course, the points $(\beta_A, \mathrm{E}(R_A))$, $(\beta_B, \mathrm{E}(R_B))$, and $(\beta_C, \mathrm{E}(R_C))$ all lie on the security market line.

Betas and expected returns for portfolios.

We define the beta of a portfolio p in the same way we defined the beta of an asset:

$$\beta_p = \frac{\mathrm{Cov}(R_p, R_m)}{\mathrm{Var}(R_m)}.$$

To express the portfolio's beta in terms of the betas of the individual assets, we apply our trait formulas, including this formula for covariances of sums:

$$\mathrm{Cov}\left(\sum_{i=1}^{m} X_i, \sum_{j=1}^{n} Y_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} \mathrm{Cov}\big(X_i, Y_j\big). \tag{4.19}$$

A calculation then shows that the beta of a portfolio is just the appropriate weighted average of the betas of the assets it contains:

$$\begin{aligned}\beta_p &= \frac{\mathrm{Cov}(R_p, R_m)}{\mathrm{Var}(R_m)} \\ &= \frac{\mathrm{Cov}\left(\sum_{i=1}^{n} p_i R_i, R_m\right)}{\mathrm{Var}(R_m)} \\ &= \frac{\sum_{i=1}^{n} p_i \,\mathrm{Cov}\left(R_i, R_m\right)}{\mathrm{Var}(R_m)} \\ &= \sum_{i=1}^{n} p_i \beta_i.\end{aligned}$$

And by manipulating the security market equation for individual assets (see Exercise 4.B.19), we can derive the security market equation for portfolios.

The security market equation for portfolios. $\mathrm{E}(R_p) - r = \beta_p(\mathrm{E}(R_{p^m}) - r).$

In words, the risk premium of portfolio p equals the beta of asset p times the market risk premium.

One portfolio worthy of special consideration is the market portfolio p^m. It's easy to check that the market portfolio's beta must be 1:

$$\beta_{p^m} = \frac{\mathrm{Cov}(R_{p^m}, R_{p^m})}{\mathrm{Var}(R_{p^m})} = \frac{\mathrm{Var}(R_{p^m})}{\mathrm{Var}(R_{p^m})} = 1.$$

This fact also follows directly from the security market equation for portfolios, as well as from the analysis of marginal risk from the beginning of this section.

The security market equation for portfolios has an interesting implication for the selection of mutual funds. To find a fund with a given expected return, it is enough to find one whose beta corresponds to that expected return. Does it follow that if you plan to place all of your capital in a single mutual fund, all mutual funds with the same beta are equally desirable? Certainly not. While a portfolio's beta measures its marginal risk and its expected return, it does not measure its absolute risk. Consider an aggressive investor who seeks an expected return consistent with a beta of 1.5. One portfolio our investor could select is one that places all of its weight on a single asset with a beta of 1.5. Alternatively, our investor could purchase a diverse portfolio with this beta. Because diversification reduces variance of returns, our investor is much better off choosing the latter option.

Our main point is that betas, whether of assets or of portfolios, are always measures of *marginal risk*, the risk associated with adding a little bit of an asset or portfolio to an existing diverse portfolio. Holding marginal risk is associated with high expected returns. But when we are considering a portfolio as a repository for all of our capital, we're no longer interested in marginal risk; what matters for us then is *absolute risk*, as measured by variance of returns. Once we choose a target expected return, we fix our portfolio's beta. But among portfolios with the beta we desire, we should choose one whose return variance is as low as possible.

USING BETAS TO EVALUATE MUTUAL FUND PERFORMANCE.

Given the thousands of mutual funds that exist today, determining which ones deserve your money is a challenging task. Investors often use past performance as the basis for their decisions. A fund's past success may be due to skilled management, low costs, exceptional forecasting prowess, or even just dumb luck, so using past performance to gauge future performance is tricky business (see Exercise 2.C.7). But beyond dealing with this difficulty, proper evaluations of fund performance must account for the relationship between risk and expected return that is embodied in the security market equation.

To see why, let's consider the performances of a few active funds. RS Investment Trust Internet Age Fund (RIAFX), Gartmore Small Cap C (GSXCX), and Franklin Templeton Hard Currency A (ICPHX) each earned annual rates of returns of approximately 12.5% between 2001 and 2004. Does this mean that all of these funds performed equally well?

To make this determination, we need to assess the funds' performances in the light of the security market equation. While the security market equation is stated in terms of expected returns, we use actual returns over the period of interest to evaluate fund performance.[11]

(*continued*)

[11] See, for example, Mark M. Carhart, "On Persistence in Mutual Fund Performance," *Journal of Finance* 52 (1997), 57–82.

(*continued*)

Over the three-year period of interest, the riskless rate of return on 30-day U.S. Treasury bills averaged about 2%. Using the Standard & Poor's 500 index as our proxy for the market portfolio, the average return on the market portfolio was approximately 4%. Plugging these figures into the security market equation, we see that a portfolio with a beta of β_p was expected to earn a return of approximately $.02 + \beta_p \times (.04 - .02) = .02 + .02\beta_p$. By combining this equation with estimates of beta from Standard & Poor's, we obtain predictions of each fund's return.

mutual fund	annual return	beta	predicted annual return
RIAFX	.125	2.45	.069
GSXCX	.125	.94	.039
ICPHX	.125	−.03	.019

As you can see, all three funds' returns exceeded predictions based on the security market equation. Since the three funds had the same returns but very different betas, we should be most impressed by the performance of the fund whose beta was lowest. As you might have guessed from its name, the RS Investment Trust Internet Age fund had a rather high beta; its performance exceeded expectations by $12.5\% - 6.9\% = 5.6\%$. But since the Templeton currency fund's beta was close to zero, its performance was even more impressive, exceeding predictions by $12.5\% - 1.9\% = 10.6\%$.

ESTIMATING BETAS.

How do financial professionals estimate beta? In theory, the beta of an asset has a clear definition: it is the covariance of the asset's return with the market portfolio's return, divided by the variance of the market portfolio's return. But in practice, a number of choices must be made to estimate beta. Since different analysts make different choices, they obtain different estimates from identical financial data.

For example, estimates of beta depend on how many years of return data are used to generate the estimate. A ten-year history gives us many more data points than a one-year history. But because ten years is long enough for a firm to change rather dramatically—for example, through mergers, spin-offs, and management turnover—older data may be much less relevant than current data to current performance. In addition, return data can be compiled on an annual, monthly, weekly, or daily basis; the theory provides little guidance as to the proper measurement interval. In practice, we see considerable variety in timing choices: Bloomberg uses weekly returns over two years, Standard & Poor's uses monthly returns over five years, and Value Line uses weekly returns over five years.[12]

[12] See pages.stern.nyu.edu/ adamodar/New_Home_Page/AppldCF/derivn/ch4deriv.html.

There is also considerable variation in the proxy chosen for the market portfolio. While market indices like the Standard & Poor's 500 and the Wilshire Index are common choices, more ambitious researchers build market portfolios that account for all traded assets on all the major U.S. stock exchanges. Some analysts argue that even this is not enough and create market portfolios that take foreign stocks into account. While these issues are tricky, the prestige and financial rewards that accrue to those who navigate them successfully are substantial, and a large research community devotes itself to bridging the gap between the theory and the data.[13]

CAPM: The evidence.

While mathematically attractive, the capital asset pricing model studied in this chapter is based on bold simplifying assumptions. For example, the assumption of homogeneous expectations is certainly not a correct description of actual investors' beliefs. But as we noted earlier, the theory is not intended as a literal description of reality; instead, it is an idealization of reality that is simple enough to submit to analysis, yet accurate enough to provide insights about how the world works.

So how does the theory match up with reality? In one sense, the opinions of financial professionals are quite divided on this issue. Some researchers claim that the CAPM fits the existing data as well as can be expected. Others feel that the fit is inadequate and provide alternative models that seem to capture the patterns in the data more accurately.[14]

But in another sense, the CAPM is an unmitigated success. While a number of asset pricing models each have their supporters, all of the models commonly studied are built upon the ideas first put forth in the CAPM. In particular, all of these models attempt to describe asset prices in terms of sensitivities of returns to different risk factors; while the CAPM focuses on a single risk factor, covariance with market returns, other models introduce additional factors to better explain the return data. Thus, while there are serious differences in opinion about precisely which model of asset pricing is best, nearly all current research in portfolio selection and asset pricing has its roots in the CAPM.

[13]For a textbook treatment of empirical finance, see John Y. Campbell, Andrew W. Lo., and Craig MacKinlay, *The Econometrics of Financial Markets*, Princeton University Press, 1996.

[14]The divide in opinion can be seen by contrasting the conclusions of two articles written for a top economics journal in honor of the CAPM's 40th anniversary: see André F. Perold, "The Capital Asset Pricing Model," and Eugene F. Fama and Kenneth R. French, "The Capital Asset Pricing Model: Theory and Evidence," both in *Journal of Economic Perspectives* 18 (3), Summer 2004.

KEY TERMS AND CONCEPTS

risk (p. 1)
returns (p. 2)
portfolio (p. 2)
mean vector (p. 5)
covariance matrix (p. 5)
efficient set (p. 5)
mixed portfolio (p. 9)
Sharpe ratio (p. 10)
capital market line (p. 12)
tangent point (p. 12)
tangent portfolio (p. 12)
market portfolio (p. 18)
marginal risk (p. 20)
hedge (p. 20)
absolute risk (p. 20)
beta (p. 20)
security market equation (p. 23)
risk premium (p. 23)
expected excess return (p. 23)
market risk premium (p. 23)
security market line (p. 23)

Exercises

Exercises 4.B.1–4.B.4 require data from the table below, which contains historical information about the means, variances, and covariances of the returns of six stocks: General Motors (GM), Anheuser-Busch (BUD), Microsoft (MSFT), McDonald's (MCD), Southwest Airlines (LUV), and Nike (NKE). (In the variance-covariance matrix, the diagonal entries are the variances of the returns of the corresponding stocks, and the off-diagonal entries are the covariances of the returns of the corresponding stock pairs.)

means:

GM	BUD	MSFT	MCD	LUV	NKE
.1112	.2639	.4082	.1735	.3464	.3298

variance and covariances:

	GM	BUD	MSFT	MCD	LUV	NKE
GM	.09584	.00419	.04337	.07601	.04606	−.00864
BUD	.00419	.04614	.08778	.02804	.01121	.06719
MSFT	.04337	.08778	.27722	.07321	.01379	.14778
MCD	.07601	.02804	.07321	.13538	.03418	.04821
LUV	.04606	.01121	.01379	.03418	.24008	−.02181
NKE	−.00864	.06719	.14778	.04821	−.02181	.23953

Exercise 4.B.1. Use the `three_assets.xlsx` workbook to answer the following questions about portfolios consisting of GM, BUD, and MSFT.

a. Determine the expected return and standard deviation of returns for the portfolio of $p = (.3, .3, .4)$.

b. Which portfolio generates the same expected returns as $p = (.3, .3, .4)$ with the lowest possible standard deviation or returns?
c. Suppose you desire an expected return of .20. Which portfolio achieves this expected return with the lowest standard deviation of returns?

Exercise 4.B.2. Use the `three_plus_one.xlsx` and `security_market_line.xlsx` workbooks to answer the following questions about portfolios consisting of GM, BUD, and MSFT, and a risk-free asset with a rate of return of .05.

a. Find a mixed portfolio that generates the same expected return as $p = (.3, .3, .4)$ but with the lowest variance of returns. Compare your answer here to your answer in Exercise 4.B.1(a).
b. Identify the tangent portfolio. What is its Sharpe ratio?
c. What is the market portfolio?
d. What are the betas of GM, BUD, and MSFT? Verify that each of the assets satisfies the security market equation.

Exercise 4.B.3. Use the `three_assets.xlsx` workbook to answer the following questions about portfolios consisting of MCD, LUV, and NKE.

a. Determine the expected returns and standard deviation of returns for the portfolio $p = (.1, .4, .5)$.
b. Which portfolio generates the same expected return as $p = (.1, .4, .5)$ with the lowest variance of returns?
c. Suppose you desire an expected return of .25. Which portfolio achieves this expected return with the lowest variance of returns?

Exercise 4.B.4. Use the `three_plus_one.xlsx` and `security_market_line.xlsx` workbooks to answer the following questions about portfolios consisting of MCD, LUV, NKE, and a riskless asset with a rate of return of .05.

a. Find a mixed portfolio that generates the same expected return as $p = (.1, .4, .5)$ but with the lowest variance of returns. Compare your answer here to your answer in Exercise 4.B.3(a).
b. Identify the tangent portfolio. What is its Sharpe ratio?
c. What is the market portfolio?
d. What are the betas of MCD, LUV, and NKE? Verify that each of the assets satisfies the security market equation.

Exercise 4.B.5. Three investors allocate their capital between the risk-free asset and the market portfolio. Which investor is the most risk averse? Which is the least risk averse?

investor	\$ in risk-free asset	\$ in market portfolio
Xiang	20,000	60,000
Yasmine	−5,000	105,000
Zoe	10,000	10,000

Exercise 4.B.6.

a. Suppose that the risk-free rate of return is 6% and the expected market rate of return is 14%. What is the return of an asset whose beta is .75?
b. Suppose that an asset with a beta of 1.5 has a rate of return of 16%, and that the expected market rate of return is 12%. What is the risk-free rate of return?

Exercise 4.B.7. Show that the beta of the risk-free asset is 0.

Exercise 4.B.8. The joint distribution of the returns of Acme Inc. and the returns on the market portfolio are presented in the following table.

		R_{p^m}	
		.08	.12
R_A	.06	.50	0
	.18	.25	.25

a. What is the beta of Acme Inc.?
b. What is the risk-free rate of return?
c. Suppose I decide to invest in a portfolio that consists of Acme Inc. stock and the market portfolio in equal parts. Express the returns to this new portfolio in terms of R_A and R_{p^m}. What are the expected value and standard deviation of the returns on the new portfolio?

Exercise 4.B.9. Suppose that investors are able to invest in two risky assets and one risk-free asset. The returns of the risky assets are represented by R_1 and R_2, respectively. The variances and covariance of these returns are given by $(\sigma_1)^2 = .0025$, $(\sigma_2)^2 = .0049$, and $\sigma_{1,2} = .0007$. The market portfolio is given by $p^m = (.30, .70)$, the expected return on the market portfolio is .0880, and the risk-free rate of return is $r = .0393$.

a. What is the variance of the returns on the market portfolio?
b. What are the betas of the two risky assets?
c. What are the expected returns of the two risky assets?

Exercise 4.B.10. The following is a description of the returns on the market portfolio, the stock of Alger Inc., and the stock of Bergman Industries:

$$\mathrm{E}(R_{p^m}) = .09 \qquad \mathrm{SD}(R_{p^m}) = .08$$
$$\mathrm{SD}(R_A) = .10 \qquad \mathrm{Cov}(R_A, R_{p^m}) = .0096$$
$$\mathrm{SD}(R_B) = .20 \qquad \mathrm{Cov}(R_B, R_{p^m}) = .0016.$$

The rate of return on the risk-free asset is $r = .05$.

Suppose you invest $1000 in Alger.

a. What is the expected return on this investment?
b. What is the variance in the returns?

Now suppose you invest $1000 in Bergman.

c. What is the expected return on this investment?
d. What is the variance in the returns?
e. If you answered (a)–(d) correctly, you found that investing $1000 in Alger generated both a higher expected return and a lower variance of returns than investing the same amount of money in Bergman. Thus, it appears that Alger has both a higher expected return and a lower level of risk than Bergman. Does this contradict the CAPM? Why or why not?

Exercise 4.B.11. The following is a description of the returns on stock in the Arvus Company and the returns on stock in Biosolve Inc.:

$$E(R_A) = .10 \qquad E(R_B) = .12$$
$$Var(R_A) = .0324 \qquad Var(R_B) = .0400$$
$$Cov(R_A, R_B) = .0306.$$

The market portfolio has expected returns of .14 and variance of returns of .0144, and the risk-free rate of return is .04.

a. Compute the expected returns and variance of returns of a portfolio that consists of 20% Arvus stock and 80% Biosolve stock.
b. Is there a portfolio consisting solely of Arvus and Biosolve that exhibits lower absolute risk than both the portfolio consisting solely of Arvus and the portfolio consisting solely of Biosolve? Why or why not?
c. Compute $Cov(R_A, R_{p^m})$ and $Cov(R_B, R_{p^m})$.
d. Which stock, Arvus or Biosolve, exhibits greater absolute risk? Why?
e. Which stock, Arvus or Biosolve, exhibits greater marginal risk? Why?

Exercise 4.B.12. Three assets numbered 1, 2, and 3 are available for investment. Their standard deviations and correlations are given by:

$$\sigma_1 = .06 \qquad \sigma_2 = .10 \qquad \sigma_3 = .15$$
$$\rho_{1,2} = .8 \qquad \rho_{1,3} = 0 \qquad \rho_{2,3} = -.2.$$

There is also a risk-free asset with a return of 5.000%, and the expected return on the market portfolio is 8.698%. The value of shares outstanding of the three assets is as follows: asset 1, $4,723,000; asset 2, $3,100,000; asset 3, $177,000. The CAPM allows us to determine the betas and expected returns of the three assets using the information above.

a. Compute the three covariances $\sigma_{1,2}$, $\sigma_{1,3}$, and $\sigma_{2,3}$.
b. Determine the market portfolio.
c. Compute the variance of the return on the market portfolio.

d. Compute the beta of each security.
e. Compute the expected return of each security.
f. Asset 3 has the highest standard deviation of the three securities, and if you did your calculations correctly, you have discovered that the expected return of asset 3 is below the risk-free rate. Explain why asset 3 is still in the market portfolio (i.e. why investors want to hold asset 3).

Exercise 4.B.13. The variances and covariances of the returns on Allston, Beryl, and Cramton stock are given in the covariance matrix below:

$$\begin{pmatrix} \sigma_A^2 & \sigma_{A,B} & \sigma_{A,C} \\ \sigma_{B,A} & \sigma_B^2 & \sigma_{B,C} \\ \sigma_{C,A} & \sigma_{C,B} & \sigma_C^2 \end{pmatrix} = \begin{pmatrix} .0064 & .001 & -.0012 \\ .001 & .01 & .0036 \\ -.0012 & .0036 & .04 \end{pmatrix}.$$

The risk-free rate of return is .05. The market portfolio is $p^m = (.17, .57, .26)$, and the expected return on this portfolio is .318.

a. What is the variance of the returns on the market portfolio?
b. A friend suggests that the market portfolio is actually an alternative portfolio whose expected return is .28 and whose variance of returns is .0063. Show that this alternative portfolio cannot be the market portfolio.
c. What are the expected returns of each of the three stocks?

Exercise 4.B.14. Suppose that last year, the risk-free rate of return was 2% and the expected market return was 8%. The table below presents the betas and realized returns for three mutual funds. By how much did each fund outperform expectations?

mutual fund	beta	realized return
RSTX	.25	.05
BCDX	1.60	.14
EFGX	2.55	.16

Exercise 4.B.15. John has $20,000 to invest. He knows that the risk-free rate of return is 3% and that the expected return on the market portfolio is 11%.

a. If John wants an expected return of 9%, how many dollars should he invest in the risk-free asset?
b. If John wants an expected return of 15%, what should he do?

Exercise 4.B.16. Judd must choose between investing all of his money in Acme Enterprises and investing all of his money in Barton Inc. Both of these companies' stocks have betas of 1.25. Does this mean that Judd's two investment options are equally good? Explain.

Exercise 4.B.17. Show that when there are two risky assets and a riskless asset, and the correlation between R_1 and R_2 is not 1 or -1, the weight on asset 1 in the tangent portfolio is

$$p_1^t = \frac{\sigma_{12}(\mu_2 - r) - \sigma_2^2(\mu_1 - r)}{\sigma_{12}(\mu_1 + \mu_2 - 2r) - \sigma_1^2(\mu_2 - r) - \sigma_2^2(\mu_1 - r)}.$$

(Hint: Use the fact that $p_2 = 1 - p_1$ to express SR_p in terms of p_1. Then, using calculus, maximize SR_p with respect to p_1 to obtain p_1^t.)

Exercise 4.B.18. This exercise works through a derivation of the security market equation. Let p^ε be the portfolio that puts weight ε on asset i and weight $1 - \varepsilon$ on portfolio p. Since

$$R_{p^\varepsilon} = \varepsilon R_i + (1 - \varepsilon) R_p,$$

it is easy to check that

$$\mathrm{E}(R_{p^\varepsilon}) = \varepsilon \mathrm{E}(R_i) + (1 - \varepsilon)\mathrm{E}(R_p) \quad \text{and}$$
$$\mathrm{Var}(R_{p^\varepsilon}) = \varepsilon^2 \,\mathrm{Var}(R_i) + (1 - 2\varepsilon + \varepsilon^2)\,\mathrm{Var}(R_p) + (2\varepsilon - 2\varepsilon^2)\,\mathrm{Cov}(R_i, R_p).$$

a. Suppose that asset i has a beta of 1, and let $p = p^m$ be the market portfolio. Show that in this case, the equations above become

$$\mathrm{E}(R_{p^\varepsilon}) = \varepsilon \mathrm{E}(R_i) + (1 - \varepsilon)\mathrm{E}(R_{p^m}) \quad \text{and}$$
$$\mathrm{Var}(R_{p^\varepsilon}) = \varepsilon^2 \,\mathrm{Var}(R_i) + (1 - \varepsilon^2)\,\mathrm{Var}(R_{p^m}),$$

and that this implies that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathrm{Var}(R_{p^\varepsilon})\right|_{\varepsilon=0} = 0.$$

b. Suppose again that $\beta_i = 1$. Argue that if $\mathrm{E}(R_i) > \mathrm{E}(R_{p^m})$, then the portfolio p^ε with ε positive and small has a higher expected return than the market portfolio and essentially the same variance as the market portfolio, contradicting the fact that the market portfolio is efficient. (More precisely, it contradicts the fact that the market portfolio sits on the capital market line, which contains all efficient (σ, μ) points, and the fact that this line has a finite slope.) Show that if $\mathrm{E}(R_i) < \mathrm{E}(R_{p^m})$, we can reach the same contradiction by choosing $\varepsilon < 0$. This proves that $\mathrm{E}(R_i) = \mathrm{E}(R_{p^m})$ whenever asset i has a beta of 1, and hence that such assets satisfy the security market equation.

c. Show that any asset i with $\beta_i \neq 0$ satisfies the security market equation. (Hint: Apply part (b) to a portfolio that puts weight $\frac{1}{\beta_i}$ on asset i and weight $(1 - \frac{1}{\beta_i})$ on the risk-free asset.)

d. Show that any asset i with $\beta_i = 0$ satisfies the security market equation. (Hint: Consider a 50/50 mix of this asset with the market portfolio and apply part (c).)

Exercise 4.B.19. Derive the security market equation for portfolios from the security market equation for individual assets. (Hint: Multiply the security market equation for each asset i by that asset's weight in the market portfolio and then sum over all assets.)

Exercise 4.B.20. It turns out that we can use the `three_plus_one.xlsx` worksheet to study models with two risky assets and a riskless asset. How? By including a third risky asset that no investor would want to hold in any positive or negative quantity.

Consider a model that includes a riskless asset with return r. Suppose that asset i has mean return $\mu_i = r$, a positive variance of returns, and a zero covariance of returns with all other risky assets. Show that it is never optimal for an investor to hold this asset in any positive or negative quantities, because he is always better off replacing asset i with the riskless asset.